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Rathkjen, Arne

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1. INTRODUCTION

When a material is anisotropic but possesses some kind of symmetry its constitutive equation, failure criterion etc. is most conveniently expressed in a coordinate system coinciding with the symmetry axes of the material.

Examples of failure criteria given in such systems are found in [1]. Quite often, however, stresses are computed in another system and in several instances it is appropriate to transform the failure criterion to this off-axes system. In other instances of course, it is appropriate to transform the stresses, but this is of no concern here. The purpose of this paper is to show how failure criteria are developed along the same lines as in [1] with the exception that the reinforcement is not placed along the coordinate axes used.

In section 2 the common background from [1] is reviewed, in section 3, 4, and 5 failure criteria are developed for materials reinforced in respectively one direction, any number of directions and in two symmetrical directions. For the sake of relative simplicity only plane stress is considered.

2. PRELIMINARIES

The failure criterion for the isotropic matrix-material is assumed to be a complete polynomial of the second degree. As shown in [1], equation (12), such a criterion may be expressed as

$$f = \frac{4S^2-CT}{4CTS^2}\left(m_{xx}+m_{yy}+\frac{2(C-T)S^2}{4S^2-CT}\right)^2+\frac{1}{4S^2}(m_{xx}-m_{yy})^2$$

$$+\frac{1}{S^2}m_{xy}-\frac{(C+T)^2S^2-C^2T^2}{(4S^2-CT)CT}=0 \tag{1}$$

when only plane stress is considered.

In (1) C, T, and S are strengths in uniaxial compression, in uniaxial tension and in pure shear respectively, and m_{xx}, m_{yy}, m_{xy} are stresses in the matrix. The condition $CT < 4S^2$ is satisfied so that the material cannot resist infinite stresses in plane stress. As a consequence the failure surface in a m_{xx}, m_{yy}, m_{xy} – coordinate system is a closed surface, an ellipsoid.

With

$$\overline{CT} = \frac{(C+T)^2S^2-C^2T^2}{4S^2-CT}$$

$$Q^2 = \frac{S^2}{CT}\,\overline{CT} \Rightarrow \frac{4Q^2-\overline{CT}}{\overline{CT}} = \frac{4S^2-CT}{CT} \tag{2}$$

the failure criterion (1) becomes

$$\frac{4Q^2-\overline{CT}}{4\overline{CT}\,Q^2}(m_{xx}+m_{yy}+\frac{2(C-T)S^2}{4S^2-CT})^2+\frac{1}{4Q^2}(m_{xx}-m_{yy})^2+\frac{1}{4Q^2}(2m_{xy})^2=1 \qquad (3)$$

or, for convenience

$$\frac{1}{4A^2}(m_{xx}+m_{yy}+2B)^2+\frac{1}{4Q^2}(m_{xx}-m_{yy})^2+\frac{1}{4Q^2}(2m_{xy})^2=1 \qquad (4)$$

with

$$A^2=\frac{\overline{CT}\,Q^2}{4Q^2-\overline{CT}}\,,\qquad B=\frac{(C-T)S^2}{4S^2-CT} \qquad (5)$$

The stress in the composite σ_{kl} is expressed in terms of the matrix stress m_{kl}, the extra stress $t_{kl}^{(n)}$ in the reinforcement in the n'th direction and the volume fraction $\varphi^{(n)}$ as

$$\sigma_{kl}=m_{kl}+\sum_n \varphi^{(n)}t_{kl}^{(n)} \equiv m_{kl}+\sum_n (\varphi t_{kl})^{(n)} \qquad (6)$$

Approximating the extra stress t_{kl} with a uniaxial tension in the direction a_m of the reinforcement gives

$$t_{kl}^{(n)}=(ta_k a_l)^{(n)} \qquad (7)$$

so that

$$m_{kl}=\sigma_{kl}-\sum_n (\varphi t a_k a_l)^{(n)} \qquad (8)$$

The extra stress is limited by

$$-t_c^{(n)}\leqq t^{(n)}<t_t^{(n)} \qquad (9)$$

where t_c and t_t are maximum extra stresses in the reinforcement in compression and tension respectively.

3. UNI-DIRECTIONAL REINFORCEMENT.

With reinforcement in one direction only, the direction cosines a_m are

$$a_x=\cos\theta\,,\qquad a_y=\sin\theta \qquad (10)$$

and the matrix stresses are

$$
\begin{aligned}
m_{xx} &= \sigma_{xx} - \varphi t \cos^2\theta \\
m_{yy} &= \sigma_{yy} - \varphi t \sin^2\theta \\
m_{xy} &= \sigma_{xy} - \varphi t \sin\theta \cos\theta
\end{aligned}
\tag{11}
$$

The stress combinations to be used in the failure criterion (4) are

$$
\begin{aligned}
m_{xx} + m_{yy} &= \sigma_{xx} + \sigma_{yy} - \varphi t \\
m_{xx} - m_{yy} &= \sigma_{xx} - \sigma_{yy} - \varphi t(\cos^2\theta - \sin^2\theta) \\
&= \sigma_{xx} - \sigma_{yy} - \varphi t \cos 2\theta \\
2m_{xy} &= 2\sigma_{xy} - 2\varphi t \sin\theta \cos\theta \\
&= 2\sigma_{xy} - \varphi t \sin 2\theta
\end{aligned}
\tag{12}
$$

and with these expressions the criterion becomes

$$
\frac{1}{A^2}(\sigma_{xx} + \sigma_{yy} + 2B - \varphi t)^2 + \frac{1}{4Q^2}(\sigma_{xx} - \sigma_{yy} - \varphi t \cos 2\theta)^2 + \frac{1}{4Q^2}(2\sigma_{xy} - \varphi t \sin 2\theta)^2 = 1 \tag{13}
$$

When t is a constant, $t = -t_c$ or $t = t_t$, (13) represents an ellipsoid in stress space. For either value of t only a semi-ellipsoid is part of the criterion. The centres of the semi-ellipsoids are located at $(\sigma_{xx} + \sigma_{yy} + 2B, \sigma_{xx} - \sigma_{yy}, 2\sigma_{xy}) =$
$-\varphi t_c(1, \cos 2\theta, \sin 2\theta)$ and
$\varphi t_t(1, \cos 2\theta, \sin 2\theta)$ respectively.

For t-values between the two limits the actual value is determined from $\partial f / \partial t = 0$ resulting in

$$
\begin{aligned}
\varphi t &= \frac{1}{A^2 + Q^2}\left(Q^2(\sigma_{xx} + \sigma_{yy} + 2B) + A^2(\sigma_{xx} - \sigma_{yy})\cos 2\theta + 2A^2\sigma_{xy}\sin 2\theta\right) \\
&= \sigma_{xx} + \sigma_{yy} + \frac{2}{A^2 + Q^2}\left(Q^2 B - A^2(\sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta)\right) \\
&= I + 2B - \frac{2A^2}{A^2 + Q^2}(\sigma_{\theta\theta} + B)
\end{aligned}
\tag{14}
$$

where $I = \sigma_{xx} + \sigma_{yy}$ is the first invariant of the stress tensor and $\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta$ is the normal stress in the direction perpendicular to the reinforcement.

With the value of φt from (14) the failure criterion (13) becomes

$$Q^2(\sigma_{xx}+\sigma_{yy}+2B)^2+A^2(\sigma_{xx}-\sigma_{yy})^2+A^2(2\sigma_{xy})^2-\frac{1}{A^2+Q^2}\times$$

$$\times(Q^2(\sigma_{xx}+\sigma_{yy}+2B)+A^2(\sigma_{xx}-\sigma_{yy})\cos 2\theta+2A^2\sigma_{xy}\sin 2\theta)^2=$$

$$=4A^2Q^2 \qquad (15)$$

Using the coordinate transformation

$$\begin{bmatrix}\sigma_{xx}+B\\ \sigma_{yy}+B\\ \sqrt{2}\,\sigma_{xy}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1+\cos 2\theta & 1-\cos 2\theta & -\sqrt{2}\sin 2\theta\\ 1-\cos 2\theta & 1+\cos 2\theta & \sqrt{2}\sin 2\theta\\ \sqrt{2}\sin 2\theta & -\sqrt{2}\sin 2\theta & 2\cos 2\theta\end{bmatrix}\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix} \qquad (16)$$

the criterion (15) is

$$\frac{p_2^2}{A^2+Q^2}+\frac{p_3^2}{2Q^2}=1 \qquad (17)$$

showing that this part of the failure surface is an elliptical cylinder with generators parallel with the p_1-axis.

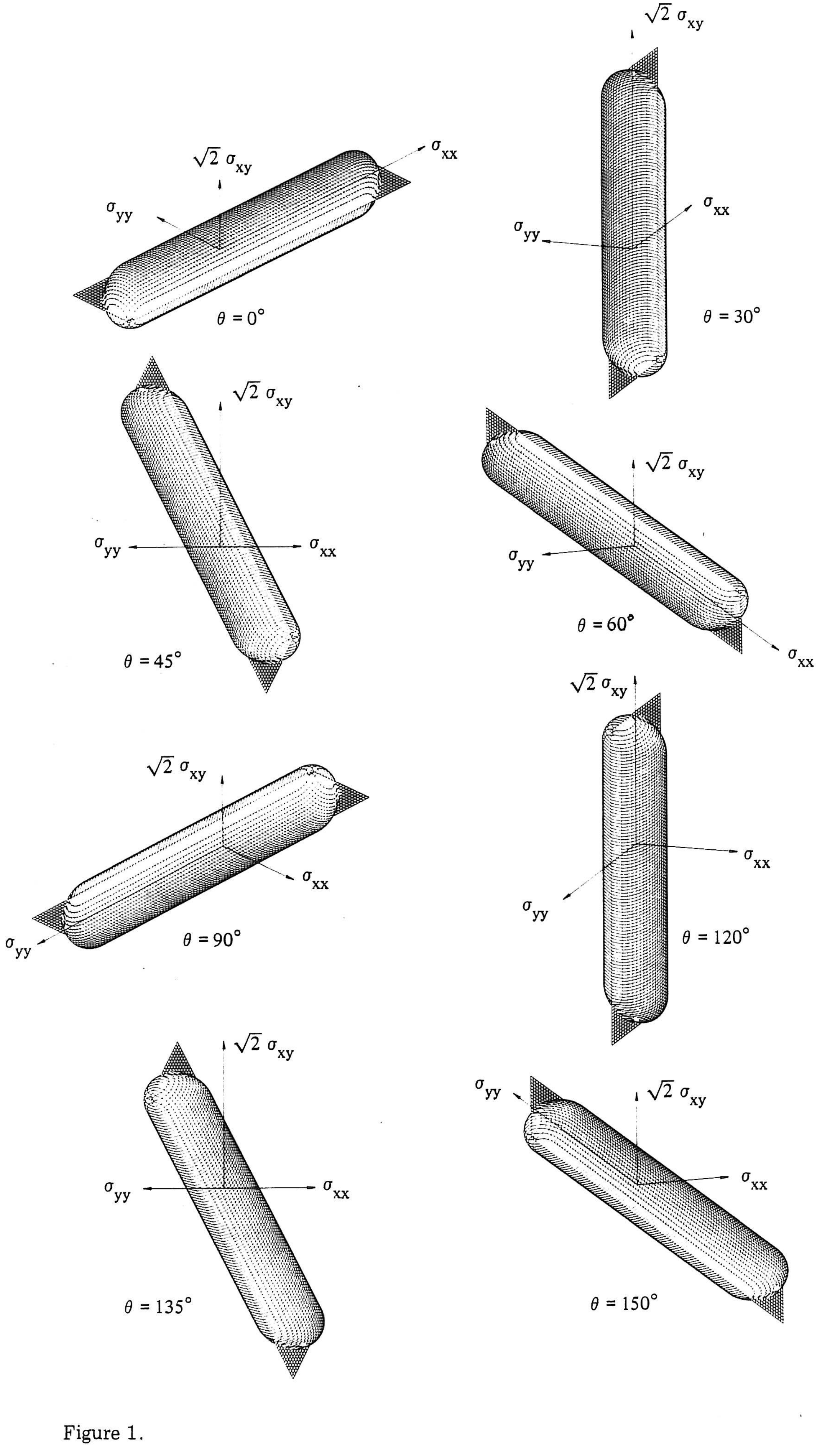
The p_1-axis in turn is parallel with the straight line connecting the centres of the semi-ellipsoids corresponding to constant values of t.

In p_1, p_2, p_3– coordinates the criterion (13) is written

$$\frac{1}{4A^2}(p_1+p_2-\varphi t)^2+\frac{1}{4Q^2}(p_1-p_2-\varphi t)^2+\frac{1}{2Q^2}p_3^2=1 \qquad (18)$$

The coordinate transformation (16) is a rotation about $(p_1, p_2, p_3)=\frac{\sqrt{2}}{2}(1, 1, 0)$ and the angle of rotation χ is $\chi=-2\theta$.

In figure 1 failure surfaces corresponding to some values of θ are shown. The most significant difference between failure surfaces corresponding to values of θ different from 0^o or 90^o and surfaces corresponding to these two values is that only with θ equal to 0^o or 90^o is the surface symmetric with respect to the σ_{xx}, σ_{yy}–plane.



$\sqrt{2}\ \sigma_{xy}$
σ_{xx}
σ_{yy}
$\theta = 0°$
$\theta = 30°$
$\theta = 45°$
$\theta = 60°$
$\theta = 90°$
$\theta = 120°$
$\theta = 135°$
$\theta = 150°$

Figure 1.

4. REINFORCEMENT IN SEVERAL DIRECTIONS

Taking a matrix reinforced in several directions the direction cosines are denoted

$$a_x^{(n)} = \cos\theta^{(n)}, \quad a_y^{(n)} = \sin\theta^{(n)} \tag{19}$$

The total volume ratio of the reinforcement is φ and for each direction the fraction is

$$\varphi^{(n)} = c^{(n)}\varphi \tag{20}$$

so that

$$\varphi^{(n)} t_{kl}^{(n)} = \varphi(cta_k a_l)^{(n)} \tag{21}$$

Matrix stresses now are

$$\begin{aligned} m_{xx} &= \sigma_{xx} - \varphi \sum_n (ct\cos^2\theta)^{(n)} \\ m_{yy} &= \sigma_{yy} - \varphi \sum_n (ct\sin^2\theta)^{(n)} \\ m_{xy} &= \sigma_{xy} - \varphi \sum_n (ct\sin\theta\cos\theta)^{(n)} \end{aligned} \tag{22}$$

and the stress combinations to be used in the failure criterion (4) are

$$\begin{aligned} m_{xx} + m_{yy} &= \sigma_{xx} + \sigma_{yy} - \varphi \sum_n (ct)^{(n)} \\ m_{xx} - m_{yy} &= \sigma_{xx} - \sigma_{yy} - \varphi \sum_n (ct\cos 2\theta)^{(n)} \\ 2m_{xy} &= 2\sigma_{xy} - \varphi \sum_n (ct\sin 2\theta)^{(n)} \end{aligned} \tag{23}$$

With these expressions the criterion becomes

$$\frac{1}{4A^2}(\sigma_{xx} + \sigma_{yy} + 2B - \varphi \sum_n (ct)^{(n)})^2 + \frac{1}{4Q^2}(\sigma_{xx} - \sigma_{yy} - \varphi \sum_n (ct\cos 2\theta)^{(n)})^2$$

$$+ \frac{1}{4Q^2}(2\sigma_{xy} - \varphi \sum_n (ct\sin 2\theta)^{(n)})^2 = 1 \tag{24}$$

Maximum strength is determined from the equations

$$\partial f / \partial t^{(n)} = 0 \tag{25}$$

to be used only when the extra stresses found are in the intervals

$$-t_c^{(n)} \leqq t^{(n)} \leqq t_t^{(n)} \tag{26}$$

As the number of failure functions is 3^N when the number of reinforcement directions is N only a small number of directions can be treated in this way. When a matrix is reinforced in more directions, however, the purpose is often to produce an isotropic material and the failure criterion for such a material does not have to be found in this way.

5. REINFORCEMENT IN TWO DIRECTIONS

As an example of a material reinforced in two directions equal amounts of reinforcement in two symmetrical directions as shown in figure 2 is taken.

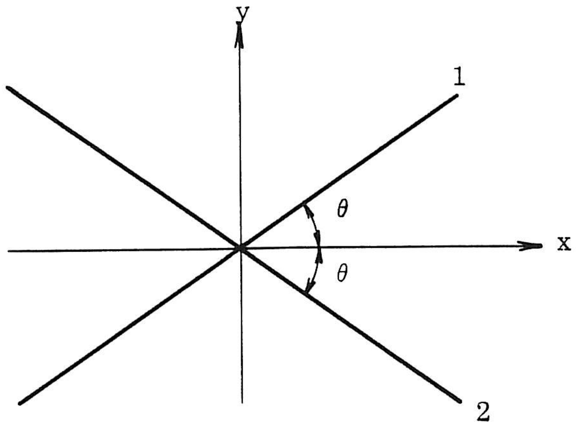


Figure 2.

In this case we have

$$c_1 = c_2 = \frac{1}{2}$$

$$\theta_1 = -\theta_2 = \theta \qquad (27)$$

$$t_{c1} = t_{c2} = t_c, \; t_{t1} = t_{t2} = t_t$$

and

$$m_{xx} + m_{yy} = \sigma_{xx} + \sigma_{yy} - \frac{1}{2}\varphi(t_1 + t_2)$$

$$m_{xx} - m_{yy} = \sigma_{xx} - \sigma_{yy} - \frac{1}{2}\varphi(t_1 + t_2)\cos 2\theta \qquad (28)$$

$$2m_{xy} = 2\sigma_{xy} - \frac{1}{2}\varphi(t_1 - t_2)\sin 2\theta$$

The failure criterion thus becomes

$$4A^2Q^2f = Q^2(\sigma_{xx} + \sigma_{yy} + 2B - \frac{1}{2}\varphi(t_1 + t_2))^2$$

$$+ A^2(\sigma_{xx} - \sigma_{yy} - \frac{1}{2}\varphi(t_1 + t_2)\cos 2\theta)^2$$

$$+ A^2(2\sigma_{xy} - \frac{1}{2}\varphi(t_1 - t_2)\sin 2\theta)^2 - 4A^2Q^2 = 0 \qquad (29)$$

Maximum strength is found from

$\partial f/\partial t_1 = 0$ and $\partial f/\partial t_2 = 0$ giving

$$\frac{1}{2}\varphi(Q^2 + A^2)\,t_1 = Q^2(\sigma_{xx} + \sigma_{yy} + 2B) + A^2(\sigma_{xx} - \sigma_{yy})\cos 2\theta$$

$$+ A^2 2\sigma_{xy}\sin 2\theta - \frac{1}{2}\varphi(Q^2 + A^2\cos 4\theta)t_2 = \frac{1}{2}\varphi(Q^2 + A^2)T_1 \qquad (30)$$

$$\frac{1}{2}\varphi(Q^2 + A^2)t_2 = Q^2(\sigma_{xx} + \sigma_{yy} + 2B) + A^2(\sigma_{xx} - \sigma_{yy})\cos 2\theta$$

$$- A^2 2\sigma_{xy}\sin 2\theta - \frac{1}{2}\varphi(Q^2 + A^2\cos 4\theta)t_1 = \frac{1}{2}\varphi(Q^2 + A^2)T_2$$

where the extra stresses are limited by

$$-t_c \leqq t_1 \leqq t_t$$

$$-t_c \leqq t_2 \leqq t_t \qquad (31)$$

The nine failure functions f_α, $\alpha = 1, 2 \ldots 9$, are found from (29) using values of t_1 and t_2 from table 1.

α	t_1	t_2	
1	$-t_c$	$-t_c$	
2	$-t_c$	T_2	
3	$-t_c$	t_t	
4	T_1	$-t_c$	
5	T_1	T_2	T_1 and T_2 from (30)
6	T_1	t_t	
7	t_t	$-t_c$	
8	t_t	T_2	
9	t_t	t_t	

Table 1.

When both extra stresses are constants (α = 1, 3, 7, and 9) the corresponding parts og the failure surface are ellipsoids, when one extra stress is constant and the other varies according to (30) (α = 2, 4, 6, and 8) the parts of the failure surface are cylinders. When both extra stresses vary according to (30) (α = 5) their values are given by

$$\varphi t_1 = \frac{Q^2(\sigma_{xx} + \sigma_{yy} + 2B) + A^2(\sigma_{xx} - \sigma_{yy})\cos 2\theta}{Q^2 + A^2\cos^2 2\theta} + \frac{2\sigma_{xy}}{\sin 2\theta}$$

$$\varphi t_2 = \frac{Q^2(\sigma_{xx} + \sigma_{yy} + 2B) + A^2(\sigma_{xx} - \sigma_{yy})\cos 2\theta}{Q^2 + A^2\cos^2 2\theta} - \frac{2\sigma_{xy}}{\sin 2\theta} \qquad (32)$$

and the corresponding parts of the failure surface are the planes

$$4f_5 = \frac{((\sigma_{xx} + \sigma_{yy} + 2B)\cos 2\theta - (\sigma_{xx} - \sigma_{yy}))^2}{Q^2 + A^2\cos^2 2\theta} - 4 = 0 \qquad (33)$$

The expressions (32) are not valid with $\theta = 0$. In this case t_1 and t_2 are equal since the matrix is reinforced in one direction only. The result in this case is given in section 3.

[1] Rathkjen, A.: Failure Criteria for Reinforced Materials, Bygningsstatiske Meddelelser, vol. 57, Nr. 1, 1–36, 1986.